

Space versus Time: Unimodular versus Non-Unimodular Projective Ring Geometries?

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Abstract

Finite projective (lattice) geometries defined over rings instead of fields have recently been recognized to be of great importance for quantum information theory. We believe that there is much more potential hidden in these geometries to be unleashed for physics. There exist specific rings over which the projective spaces feature two principally distinct kinds of basic constituents (points and/or higher-rank linear subspaces), intricately interwoven with each other — unimodular and non-unimodular. We conjecture that these two projective “degrees of freedom” can rudimentary be associated with spatial and temporal dimensions of physics, respectively. Our hypothesis is illustrated on the projective line over the smallest ring of ternions. Both the fundamental difference and intricate connection between time and space are demonstrated, and even the ring geometrical germs of the observed macroscopic dimensionality (3+1) of space-time and the arrow of time are outlined. Some other conceptual implications of this speculative model (like a hierarchical structure of physical systems) are also mentioned.

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1 Introduction

If a theoretical physicist working on unification of quantum mechanics and general relativity is asked to pinpoint the most serious problems they face, the answer will most likely be: the interpretation of the *time* and the observed macroscopic *dimensionality of space-time*. The first problem of the two is central to any approach to quantum gravity that ascribes, by first instance, an important role to classical general relativity and stems from the fundamentally different roles played by the concept of time in quantum theory and in general relativity. In quantum theory, time does not behave as a physical quantity in the usual sense, since — unlike spatial coordinates — it is not represented by an operator, being rather treated as a background parameter employed to mark the evolution of the system. Moreover, the notion of an event happening at a given time plays a crucial role in the technical and conceptual foundations of quantum theory. Classical general relativity, however, handles time in a very different manner. Time is not regarded as a background parameter, even in the broad-minded sense of special relativity, namely as an aspect of a fixed, background space-time structure. It is rather our *interpretation* of the concept of time which makes us to either view our reality as evolving in three dimensions (“arrow” of time), or as being some kind of a four-dimensional construct (“frozen” time).

Typically, physicists simply take the observed macroscopic dimensionality of space-time for granted and do not bother why it has just four rather than any other number of dimensions. Only seldom do they embark upon speculations in this respect. Such contemplations begin, as a rule, by noticing that a given physical theory has considerably different properties in space-times of different dimensionality. By claiming that some feature/s of the theory typical of the four-dimensional space-time is/are fundamental, a reason can be offered for our space-time having the observed number of dimensions. These explorations have, apart from the quest for a rational explanation of the dimensionality of the physical space-time, also a more practical aspect. Quantum field theories are notorious for being ill defined in four-dimensional space-time but can often be seen to fare well in space-times of a different dimensionality. Successes of quantum field theory in both higher- and lower-dimensional space-times almost lead to an ironical statement that the only reason for

our universe to be endowed with four dimensions is the irrationality of such a choice: the four-dimensional space-time seems the most problematic setting for a quantum field theory to work. Among further physically-based arguments, it is worth mentioning Weyl's well-known observation that the Maxwell equations are tied uniquely to the $3+1$ space-time, and/or intriguing Ehrenfest's reasoning that stable atoms are only possible in $3+1$ dimensions. Another class of well-known heuristic inquiries is more mathematically oriented. Here we can, for example, rank the fact that the Weyl tensor, which in Einstein's gravitation theory carries information about that part of the space-time curvature which is not locally determined by the energy-momentum, vanishes in less than four dimensions, or a topological reason that $n \neq 4$ dimensional manifolds always feature a unique differentiable structure, while those with $n = 4$ do not. Finally, there is a large and still growing group of scholars who favour the so-called anthropic principle for a rational explanation of the macro-dimensionality of space-time.

It may well be that the two above-mentioned problems of quantum gravity, viz. the "strange behaviour" of time and the fact that our Universe features just four macroscopic dimensions, are intimately linked with each other, being in fact the two sides of the same coin. This is also the point of view adopted in this paper. In what follows we shall introduce a simple mathematical model which gives a sound formal footing to such a hypothesis. The model rests on the concept of the projective line defined over a ring instead of a field (Blunck and Havlicek 2000; Veldkamp 1995; Herzer 1995; and Blunck and Herzer 2005); the principal difference between the two kinds of geometry lies in the fact that whereas in a field every non-zero element has its inverse, in a ring which is not a field there exist non-zero elements lacking inverses (and so in this case we cannot introduce the operation of division, i.e., multiplication by the inverses). Although this concept was introduced into physics only recently, it has already produced a number of crucial insights into the nature of finite-dimensional quantum systems (see, e.g., Havlicek and Saniga 2008a; Saniga, Planat and Pracna 2008; Planat and Baboin 2007, and references therein). The model proposed, and its envisaged higher-order generalizations, will serve as another illustration of our belief that there is much potential hidden in these remarkable finite geometries to be unleashed for physics.

2 Smallest Line over Ternions — the Seed of Space-Time?

We shall consider a finite associative ring with unity $1 (\neq 0)$, R , and denote the *left* module on two generators over R by R^2 . The set $R(r_1, r_2)$, defined as follows

$$R(r_1, r_2) := \{(\alpha r_1, \alpha r_2) | (r_1, r_2) \in R^2, \alpha \in R\}, \quad (1)$$

is a left *cyclic* submodule of R^2 . Any such submodule is called *free* if the mapping $\alpha \mapsto (\alpha r_1, \alpha r_2)$ is injective, i.e., if $(\alpha r_1, \alpha r_2)$ are all *distinct*. Next, we shall call $(r_1, r_2) \in R^2$ *unimodular* if there exist elements x_1 and x_2 in R such that

$$r_1 x_1 + r_2 x_2 = 1. \quad (2)$$

It can easily be shown that if (r_1, r_2) is unimodular, then $R(r_1, r_2)$ is free; any such free cyclic submodule represents a point of the projective line defined over R , $P(R)$ (Blunck and Havlicek 2000; Veldkamp 1995; Herzer 1995; and Blunck and Herzer 2005):

$$P(R) := \{R(r_1, r_2) | (r_1, r_2) \text{ unimodular}\}. \quad (3)$$

We just mention in passing that in any such geometry a point is a *set* of pairs/vectors (of cardinality $|R|$) and has thus a subtle internal structure, which is in sharp contrast with Euclid's point of view that "a point is that which has no parts." Obviously, every projective line over any ring features free cyclic submodules generated by unimodular vectors (in the sequel also called unimodular points) and over a vast majority of finite rings these are *the only* free cyclic submodules of R^2 . Yet, as we shall soon see, there are also rings which in addition yield free cyclic submodules generated by *non*-unimodular vectors. In light of this fact, and following also the spirit and strategy of Brehm, Greferath and Schmidt (1995), it is reasonable to consider a more general concept of the projective ring line, namely

$$\hat{P}(R) := \{R(r_1, r_2) | R(r_1, r_2) \text{ free}\}. \quad (4)$$

So

$$\hat{P}(R) = P(R) \cup \tilde{P}(R) \quad (5)$$

Table 1: Addition (*left*) and multiplication (*right*) in R_\diamond .

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	6	7	5	4	2	3
2	2	6	0	4	3	7	1	5
3	3	7	4	0	2	6	5	1
4	4	5	3	2	0	1	7	6
5	5	4	7	6	1	0	3	2
6	6	2	1	5	7	3	0	4
7	7	3	5	1	6	2	4	0

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	1	3	7	5	6	4
3	0	3	5	3	6	5	6	0
4	0	4	4	0	4	0	0	4
5	0	5	3	3	0	5	6	6
6	0	6	6	0	6	0	0	6
7	0	7	7	0	7	0	0	7

with $\tilde{P}(R)$ standing for the part of the projective line comprising solely the points generated by non-unimodular vectors (also referred to as non-unimodular points). The two parts of this generalized ring line, if both non-empty, are, on the one side, very different from each other, yet, on the other side, intricately interwoven with each other. And it is this relation between the two parts of the line which is central to our subsequent discussions.

In order to see this explicitly, we shall have a detailed look at such generalized projective line over the smallest ring of ternions R_\diamond , i. e. the ring defined as follows

$$R_\diamond \equiv \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in GF(2) \right\}, \quad (6)$$

where $GF(2)$ is the Galois field of two elements. This is, up to isomorphisms, the unique non-commutative ring of order eight, endowed with two invertible elements (units) and six zero-divisors, and, most interestingly, also the smallest ring where $\tilde{P}(R)$ is not an empty set (Saniga, Havlicek, Planat and Pracna 2008; Havlicek and Saniga 2008b). Employing its addition and multiplication tables (Table 1), we readily find altogether 36 *unimodular* vectors which generate 18 different free cyclic submodules, namely

$$\begin{aligned}
 R_\diamond(1,0) &= R_\diamond(2,0) = \{(0,0), (6,0), (4,0), (7,0), (5,0), (3,0), (2,0), (1,0)\}, \\
 R_\diamond(1,6) &= R_\diamond(2,6) = \{(0,0), (6,0), (4,0), (7,0), (5,6), (3,6), (2,6), (1,6)\}, \\
 R_\diamond(1,3) &= R_\diamond(2,3) = \{(0,0), (6,0), (4,0), (7,0), (5,3), (3,3), (2,3), (1,3)\}, \\
 R_\diamond(1,5) &= R_\diamond(2,5) = \{(0,0), (6,0), (4,0), (7,0), (5,5), (3,5), (2,5), (1,5)\}, \\
 R_\diamond(7,3) &= R_\diamond(4,3) = \{(0,0), (6,0), (4,0), (7,0), (0,3), (6,3), (4,3), (7,3)\}, \\
 R_\diamond(7,5) &= R_\diamond(4,5) = \{(0,0), (6,0), (4,0), (7,0), (0,5), (6,5), (4,5), (7,5)\}, \\
 R_\diamond(1,7) &= R_\diamond(2,4) = \{(0,0), (6,6), (4,4), (7,7), (5,6), (3,0), (2,4), (1,7)\}, \\
 R_\diamond(1,4) &= R_\diamond(2,7) = \{(0,0), (6,6), (4,4), (7,7), (5,0), (3,6), (2,7), (1,4)\}, \\
 R_\diamond(1,1) &= R_\diamond(2,2) = \{(0,0), (6,6), (4,4), (7,7), (5,5), (3,3), (2,2), (1,1)\}, \\
 R_\diamond(1,2) &= R_\diamond(2,1) = \{(0,0), (6,6), (4,4), (7,7), (5,3), (3,5), (2,1), (1,2)\}, \\
 R_\diamond(4,1) &= R_\diamond(7,2) = \{(0,0), (6,6), (4,4), (7,7), (0,5), (6,3), (7,2), (4,1)\}, \\
 R_\diamond(7,1) &= R_\diamond(4,2) = \{(0,0), (6,6), (4,4), (7,7), (0,3), (6,5), (4,2), (7,1)\}, \\
 R_\diamond(3,7) &= R_\diamond(3,4) = \{(0,0), (0,6), (0,4), (0,7), (3,0), (3,6), (3,4), (3,7)\}, \\
 R_\diamond(5,7) &= R_\diamond(5,4) = \{(0,0), (0,6), (0,4), (0,7), (5,0), (5,6), (5,4), (5,7)\}, \\
 R_\diamond(5,1) &= R_\diamond(5,2) = \{(0,0), (0,6), (0,4), (0,7), (5,5), (5,3), (5,2), (5,1)\}, \\
 R_\diamond(3,1) &= R_\diamond(3,2) = \{(0,0), (0,6), (0,4), (0,7), (3,5), (3,3), (3,2), (3,1)\}, \\
 R_\diamond(6,1) &= R_\diamond(6,2) = \{(0,0), (0,6), (0,4), (0,7), (6,5), (6,3), (6,2), (6,1)\}, \\
 R_\diamond(0,1) &= R_\diamond(0,2) = \{(0,0), (0,6), (0,4), (0,7), (0,5), (0,3), (0,2), (0,1)\},
 \end{aligned}$$

and six *non-unimodular* vectors giving rise to three distinct free cyclic submodules, namely

$$\begin{aligned}
 R_\diamond(4,6) &= R_\diamond(7,6) = \{(0,0), (6,0), (0,6), (6,6), (4,0), (7,0), (7,6), (4,6)\}, \\
 R_\diamond(4,7) &= R_\diamond(7,4) = \{(0,0), (6,0), (0,6), (6,6), (4,4), (7,7), (7,4), (4,7)\}, \\
 R_\diamond(6,4) &= R_\diamond(6,7) = \{(0,0), (6,0), (0,6), (6,6), (0,4), (0,7), (6,7), (6,4)\}.
 \end{aligned}$$

The structure of both the sets and how the two are intertwined can be fairly well visualised and grasped in terms of a network of broken line-segments (polygons) as depicted in Figures 1 and 2, respectively. In both figures, a circle with an attached two-digit number XY stands for the (X, Y) vector and line-joins of two circles indicate that the two corresponding vectors lie on the same free cyclic submodule; the size of vectors is roughly proportional to the number of submodules they

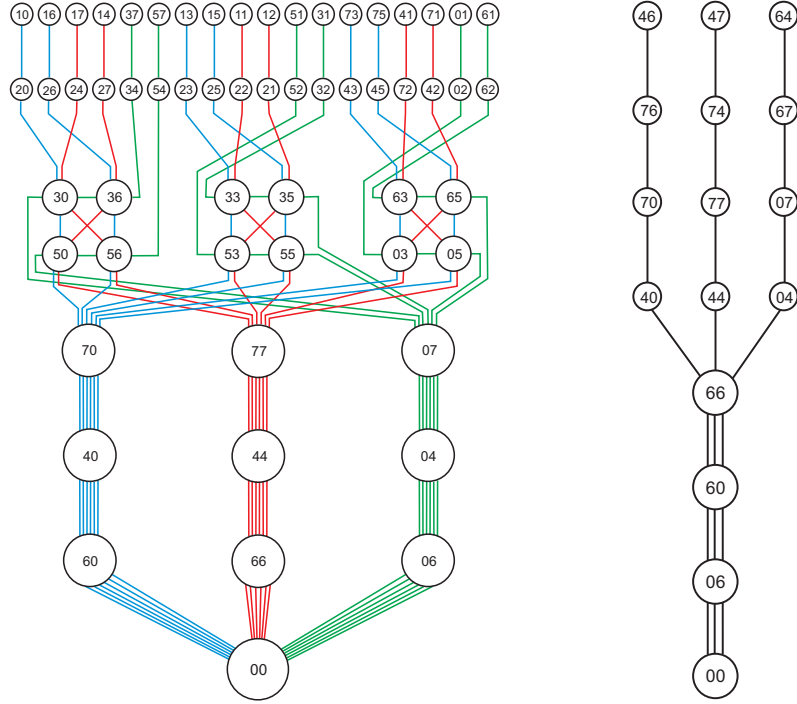


Figure 1: A diagrammatic illustration of the structure of the unimodular (*left*) and non-unimodular (*right*) parts of the projective line over the smallest ring of ternions. The symbols and notation are explained in the text.

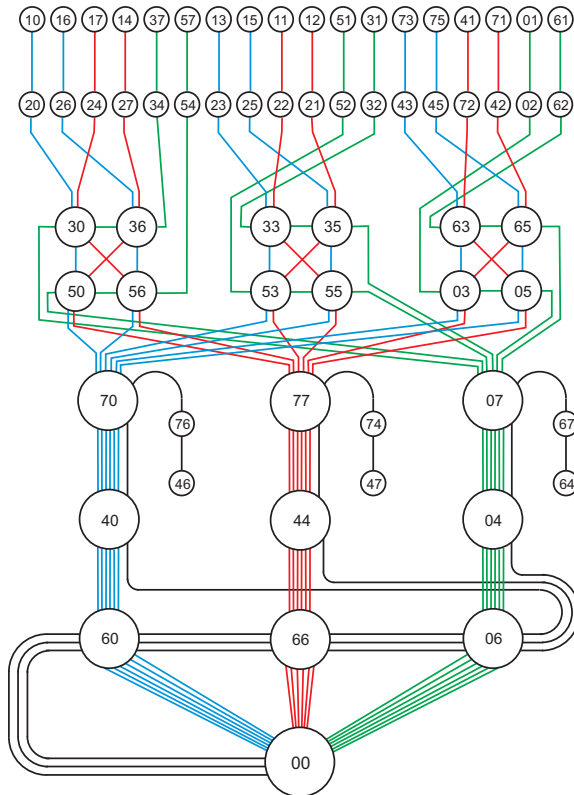


Figure 2: A diagrammatic sketch of the intricate link between the two parts of the line shown in the preceding figure.

are contained in. Just a passing look at Figure 1 reveals a principal distinction between the two “sectors/regimes” of the line.

The first fact to be easily noticed is the difference in the cardinalities of the two sets, a rather trivial issue. The second feature is a bit more intricate: whilst in the unimodular configuration the only common element of all the 18 points is the vector $(0, 0)$, the three non-unimodular points share (three) additional pairs. This latter property can be rigorously accounted for after the concepts of *neighbour/distant* are introduced (Blunck and Havlicek 2000; Veldkamp 1995; Herzer 1995; and Blunck and Herzer 2005). Given the obvious fact that the $(0, 0)$ vector belongs to every cyclic submodule, we shall call two distinct points $R(r_1, r_2)$ and $R(s_1, s_2)$ of a projective line distant if $|R(r_1, r_2) \cap R(s_1, s_2)| = 1$ and neighbour if $|R(r_1, r_2) \cap R(s_1, s_2)| > 1$. We then find that *all* non-unimodular points are pairwise *neighbour*, whereas the maximum number of mutually neighbour points in the unimodular case is $6 \neq 18$ (any point is obviously neighbour to itself). Hence, in the unimodular case it also makes sense to ask what the maximum number of pairwise *distant* points is, the answer being — *three*. These facts are illustrated in Figure 1, left, by the use of three different colours. Let us pick up one blue point, for example $R(1, 0)$. Then all the points of the same colour have the property of being pairwise neighbour. Hence, to find a distant point to our selected blue point, we have to go to the sets of different colour, say red. However, only four out of six red points are distant to our selected point; we can pick up any of them, say $R(1, 1)$. The last, third point distant to the two selected must necessarily be a green one, and one can take either $R(0, 1)$ or $R(6, 1)$. This reasoning can also be rephrased the other way round; by choosing any triple (i. e., the maximum number) of pairwise distant points, the set of unimodular points is naturally partitioned into (i. e., factored into a disjoint union of) three maximum sets of mutually neighbour points, with different triples yielding *one and the same* partitioning (the three sets distinguished by different colours in both Figure 1, left, and Figure 2). The last pronounced difference between the two sectors is perhaps most interesting and most intriguing as well. If we take any unimodular point, we see that the only vectors that are unique to the point are its two generators; that is, any other vector on each unimodular free cyclic submodule belongs also to some other submodule(s)/point(s) (see Figure 1, left). If we look at any of the three non-unimodular points (Figure 1, right), we find that apart from its two generating vectors there are other two vectors that lie on just this point. This “peculiar” feature enables the so-called “geometric condensation” phenomenon to take place in terms of which the “condensate” of our non-unimodular sub-configuration is found to be isomorphic to nothing but the ordinary projective line over $GF(2)$ (Saniga 2008); one simply associates the set of the four common vectors $\{(0, 0), (6, 0), (0, 6), (6, 6)\}$ with the $(0, 0)$ vector and the remaining three quadruples $\{(4, 0), (7, 0), (7, 6), (4, 6)\}$, $\{(4, 4), (7, 7), (7, 4), (4, 7)\}$ and $\{(0, 4), (0, 7), (6, 7), (6, 4)\}$ with the $(1, 0)$, $(1, 1)$ and $(0, 1)$ vectors of the $GF(2)$ -line, respectively.¹ When it comes to the coupling between the two parts of the line (Figure 2), another remarkable feature is encountered; *every* non-unimodular point is *neighbour* to *every* unimodular one. If we regard neighbourness between two points as a tighter link when compared to the distant relation, we can also say that the coupling between every non-unimodular and every unimodular point is more intimate than that between any mutually distant (hence unimodular) points.

At this stage the amount of information gathered about the structure of our smallest ternionic projective line is sufficient for us to make a step out of pure mathematics and show that this line may be of great relevance for physics, especially as a sort of conceptual guide for approaching the two major quantum gravity problems discussed in the introduction. To this end in view, we have first to realize that our line is rather simple as featuring only 21 points in total, and the notions of neighbour/distant are purely algebraic ones (not the slightest trace of metric whatsoever at this level!). Despite of these facts, let us nevertheless make a daring hypothesis and take this configuration as a finite prototype of space-time by tentatively identifying its unimodular part with the “seeds” of spatial degrees of freedom and its non-unimodular portion with the “buds” of time. In light of the above-discussed properties of the line, such an identification immediately entails a crucial distinction between space and time, the former being “more heterogeneous and less compact” (existence of both mutually neighbour and mutually distant unimodular points compared with only mutually neighbour non-unimodular ones) and “more complex” (the unimodular set featuring six times more elements than the non-unimodular one) than the latter. Moreover, given the unique partitioning of the unimodular aggregate of points induced by any set of three pairwise distant members, our ternionic spatial degrees of freedom are already endowed with something

¹It is also worth mentioning here that it is precisely this property that, at least in the case of rings of ternions, puts projective lines on a different footing than any higher-dimensional projective spaces (Havlicek and Saniga 2008b).

which can be regarded as a first trace of the observed three-dimensionality of space; each of the three maximum sets of mutually neighbour points viewed as the germ of a single spatial dimension. In the same spirit, complete absence of the notion of mutually distant on the non-unimodular set lends itself as a natural explanation of the observed uni-dimensionality of time and the unique geometric condensation phenomenon may well represent nothing but a ternionic “germ” of the arrow/unidirectionality of time. At this stage we cannot elaborate more on how this condensation/contraction phenomenon gives rise to the microscopic irreversibility that is required to occur in the laws of nature to explain the arrow of time. But what is already firmly established is the impossibility of making our time dimension two-directional, because that would require a sort of natural “expansion” of the non-unimodular part of our ternionic line to a (part of) projective line over a certain bigger ring, which is clearly not the case. Also, it is worth stressing that this arrow-of-time effect is strictly due to a mathematical phenomenon.

This difference between space and time becomes even more pronounced after we take into account the already-mentioned fact that rings generating both kinds of points seem to occur rather sparsely when compared to those which do not (only eight such rings out of 131 up to order 31, see below); that is, a generic projective ring line favours the universe lacking the time dimension.

3 What Is a Next Move?

In order to make the above-described space-time model more realistic, we obviously have to employ projective lines endowed with many more points and, so, consider rings of much higher orders. When doing so, two crucial, and rather severe, constraints have to be preserved: the line must be endowed with both unimodular and non-unimodular points (to “generate” both space and time) and the maximum number of mutually distant (unimodular) points must be three (to account for the observed dimensionality of space). To meet the latter constraint the necessary condition is to focus only on rings of even orders. The former constraint seems much more stringent and much more problematic to ascertain because, to our best knowledge, not only is there no general recipe for finding rings that yield both unimodular and non-unimodular free cyclic submodules, but also very little is known about the corresponding projective lines and/or higher-dimensional spaces (Brehm, Greferath and Schmidt 1995). We, therefore, decided for a case-by-case inspection of all the finite rings up to order 31 (altogether 131, see also Saniga, Planat, Kibler and Pracna 2007; Saniga, Planat and Pracna 2006) and found, apart R_\diamond , six non-isomorphic rings of order 16 (one of them being isomorphic to $GF(2) \times R_\diamond$) and a single ring of order 24 (that isomorphic to $GF(3) \times R_\diamond$) to meet our constraints; remarkably, all of them being, like R_\diamond itself, *non-commutative*. The five rings of order 16 all feature 12 zero-divisors and split into two distinct classes (A and B) differing from each other in the cardinalities of their ideals of order 2, 4, and 8 — 2, 2, 2 versus 3, 1, 2, respectively. And although this difference does not manifest itself in the unimodular sector of the associated projective lines (Saniga, Planat and Pracna 2006), in the non-unimodular sector it considerably does; for in the former class this sector comprises six points and its condensate is isomorphic to $P(Z_4)$ and/or $P(GF(2)[x]/\langle x^2 \rangle)$, whilst in the latter class it contains nine points and its condensed form is much more involved, not having counterpart in any projective ring line (Saniga 2008). In light of our hypothesis this would mean that we have two qualitatively distinct space-times having identical spatial characteristics but substantially different time dimensions. Moving to the line over $GF(2) \times R_\diamond$ we find its non-unimodular sector to condense into $P(GF(2) \times GF(2))$ and that of the line over the ring of order 24 into $P(GF(2) \times GF(3))$. Our findings are, for the reader’s convenience, summarized in Table 2.

Table 2: Basic properties of several small projective ring lines endowed with both unimodular and non-unimodular sectors. The first column lists the type of the line, the second and third columns show the cardinalities of its unimodular and non-unimodular sectors, respectively, and the last column gives the character of the associated condensate.

$\hat{P}(R_\diamond)$	18	3	$P(GF(2))$
$\hat{P}(16/12A)$	36	6	$P(Z_4)$ or $P(GF(2)[x]/\langle x^2 \rangle)$
$\hat{P}(16/12B)$	36	9	not a ring line(?)
$\hat{P}(GF(2) \times R_\diamond)$	54	9	$P(GF(2) \times GF(2))$
$\hat{P}(GF(3) \times R_\diamond)$	72	12	$P(GF(2) \times GF(3)) \simeq P(Z_6)$

These few examples give us important clues as to what one can expect when making use of higher-order ring lines. Remarkably, one finds that, except for the case where the condensate is not a ring line, the ratio between the number of unimodular and non-unimodular points remains the same (six). When focusing on condensates themselves, which practically contain all essential information about the structure of the corresponding “seed” time dimensions, we see, on the one hand, a big discrepancy even if rings are of the same order and having the same number of zero-divisors (the two 16/12 cases), but, on the other hand, also a nice hierarchic built-up principle at work: the condensate of $\tilde{P}(GF(q) \times R_\diamond)$ is obviously isomorphic to $P(GF(2) \times GF(q))$, for q being any power of a prime. Another noteworthy fact is that, with the exception of the 16/12B case, all the condensates enjoy the property of having maximum sets of pairwise distant points of cardinality three, like the unimodular parts of the parent lines.

Imagine now that the order and complexity of the underlying ring of our “amphibian” (that is, featuring both the sectors) projective line is so big that it contains a large number of subrings of various orders such that the projective lines defined over them are also “amphibians.” This means that our parent space-time will encompass a unique aggregate of “sub-space-times” of different smaller orders and complexity, linked to each other in a particularly hierarchic way that reflects the relation between the individual subrings. As we deal with finite rings, these aggregates will always have a limited number of members. It is likely that for some orders such space-time collections will be much richer than for others. Hence, our hierarchy will basically be two-fold: it will run not only within a given order (local), but through different orders as well (global), the two meeting in the not further reducible building block — our above-described ternionic space-time. And such hierarchy applies not only to the lines (space-times) as a whole, but also separately to their two sectors (space and time) and to their condensates (imprints of time’s arrows) as well. Couldn’t, then, our universe simply be a projective ring line of a huge, yet still finite order (Table 3), unjustly neglected and inadequately hidden under a variety of disguises like a differentiable pseudo-Riemannian manifold, a world of strings and branes, etc.? It would certainly be desirable to compare the mathematical language of our model with that of the two currently mainstream competing quantum gravity formalisms, namely string M-theory and loop quantum gravity (for a most recent relevant paper in this respect, see Bojowald, 2009); yet, as our familiarity with these theories is rather superficial and, therefore, insufficient to do such task properly, we leave it to the reader, if curious enough.

Table 3: Space-time viewed as the projective line over a (yet-to-be-found) large finite ring.

Space-time	Projective Ring Line of a Very Large Order
Space	Set of Unimodular Points
Time	Set of Non-Unimodular Points
3D of Space	Three Unique Maximum Sets of Mutually Neighbour Unimodulars
1D of Time	Non-Unimodulars Form One Maximum Set of Mutually Neighbours
Arrow of Time	Condensation Phenomenon

4 A Few Final Musings

Apart from the quantum gravity issues, the above-described projective ring line concept of space-time may also be relevant and taken as a starting conceptual point for addressing the behaviour of highly complex (hierarchic) systems. A usual approach starts with modelling these systems as isolated (closed) and subject to unitary evolution, i. e., symmetric to time reversal. Any emergence of the “arrow” of time is ascribed to making these systems open, that is, coupled to their surrounding. And all kinds of classical models for open systems are based on the assumption that this coupling is weak; if the coupling becomes stronger and stronger we run into serious trouble due to the requirement of the continuity of both time and space dimensions. In our approach such problems should not emerge, as at the deepest levels the hierarchy and complexity principles pertain not only the systems and their environment(s), but — as shown in the preceding section — also to their “background” space-times. A closely related question is the fine structure of hierarchical systems, in particular emergence of qualitative new kinds of coupling/bonds between subsystems. In our model, this is surmised to be accounted for by a distinguished role played by rings (and so lines and associated space-times) of certain orders, especially by power-of-two ones. Here, one can

expect a number of interesting links to various long-standing open mathematical problems (outer automorphisms of groups, (non-)existence of projective planes of composite orders, distribution of primes, the Riemann hypothesis, etc.) to emerge in various contexts.

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